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Spacetimes in Which the Ricci Equations Characterize the Riemann Tensor

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It has recently been asked whether a fourth-order tensor **K** with all the algebraic symmetries of a Riemann tensor, and which satisfies the Ricci equations (with covariant derivative constructed from the metric **g** in the usual way), is always equal to the Riemann tensor **R** of the metric **g**; and a positive answer has been given for a generic tensor **K** in any nonflat 4-dimensional spacetime. In this paper it is shown that the result is also true in a generic 4-dimensional spacetime for any nontrivial tensor **K**. In addition, those special spacetimes where the result fails are given explicitly in terms of the Petrov types of their Weyl and Plebanski tensors.

1. INTRODUCTION

A Riemann tensor $R^a{}_{bcd}$ defined in terms of a metric g_{ab} in the usual way identically satisfies

$$R^a{}_{bcd} = -R^a{}_{bdc} \tag{1a}$$

$$R^a{}_{[bcd]} = 0 \tag{1b}$$

$$g_{ai}R^i{}_{bcd} = -g_{bi}R^i{}_{acd} \tag{1c}$$

and also the Ricci equations,

$$2R^a{}_{bcd;[ef]} = -R^i{}_{bcd}R^a{}_{ief} + R^a{}_{icd}R^i{}_{bef} + R^a{}_{bid}R^i{}_{cef} + R^a{}_{bci}R^i{}_{def} \tag{2}$$

where the covariant derivative is defined in terms of the Lorentz metric g_{ab}, which is used to raise and lower indices in the usual way.

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Recently it has been asked under which circumstances a "curvature candidate" $K^a{}_{bcd}$ satisfying

$$K^a{}_{bcd} = -K^a{}_{bdc} \tag{3a}$$

$$K^a{}_{[bcd]} = 0 \tag{3b}$$

$$g_{ai}K^i{}_{bcd} = -g_{bi}K^i{}_{acd} \tag{3c}$$

and satisfying the Ricci-type equation

$$2K^a{}_{bcd;[ef]} = -K^i{}_{bcd}K^a{}_{ief} + K^a{}_{icd}K^i{}_{bef} + K^a{}_{bid}K^i{}_{cef} + K^a{}_{bci}K^i{}_{def} \tag{4}$$

is equal to the Riemann tensor $R^a{}_{bcd}$ of the metric (Rendall, 1989*a*,*b*).

This question has been answered in terms of the type of the curvature candidate $K^a{}_{bcd}$ in a 4-dimensional spacetime **M**. It has been shown that such a curvature candidate is indeed the Riemann tensor of the metric g_{ab} for a very large class of curvature candidates (Edgar, 1990), but some very specialized counter examples have been found (Rendall, 1989*b*). However, it has also been shown that this result is true for a "generic" curvature candidate, i.e., for an open dense set of curvature candidates in the Whitney C^∞ topology (Rendall, 1989*b*; Edgar, 1990).

We now wish to find whether we can make a similar statement for generic spacetimes. We use Rendall's definition of a generic spacetime as being a spacetime **M** on which there exists an open dense subset of the space of all Lorentz metrics on **M** with Whitney C^k topology (Rendall, 1988*a*). The first step is to find out explicitly for which spacetimes (defined as a class of Riemann tensors) the Ricci equations are sufficient to characterize the Riemann tensor. The next step is to determine whether there exists an open dense subset of Lorentz metrics in a Whitney C^k topology which contains only Riemann tensors of this class.

Edgar (1990) compared equation (2) to the usual Ricci equation for $K^a{}_{bcd}$,

$$2K^a{}_{bcd;[ef]} = -K^i{}_{bcd}R^a{}_{ief} + K^a{}_{icd}R^i{}_{bef} + K^a{}_{bid}R^i{}_{cef} + K^a{}_{bci}R^i{}_{def} \tag{5}$$

resulting in the algebraic constraint equation

$$0 = K^i{}_{bcd}P^a{}_{ief} - K^a{}_{icd}P^i{}_{bef} - K^a{}_{bid}P^i{}_{cef} - K^a{}_{bci}P^i{}_{def} \tag{6}$$

where

$$P^a{}_{bcd} = R^a{}_{bcd} - K^a{}_{bcd} \tag{7}$$

The set of equations (7) was split into one subset containing only $\underset{0}{K}_{ab}$ (the trace-free "Ricci part"),

$$\underset{0}{K}_{ic} P^i{}_{bef} + \underset{0}{K}_{ib} P^i{}_{cef} = 0 \tag{8}$$

and one subset containing only $\underset{0}{K}^a{}_{bcd}$ (the trace-free "Weyl part"),

$$0 = \underset{0}{K}^i{}_{bcd} P^a{}_{ief} - \underset{0}{K}^a{}_{icd} P^i{}_{bef} - \underset{0}{K}^a{}_{bid} P^i{}_{cef} - \underset{0}{K}^a{}_{bci} P^i{}_{def} \tag{9}$$

Then we applied classification schemes to $\underset{0}{K}^a{}_{bcd}$ (and $\underset{0}{K}_{ab}$) and for each class we substituted the respective canonical form into (9) [and (8)]. For some classes, $P^a{}_{bcd}$ was found to be identically zero, and so $K^a{}_{bcd}$ and $R^a{}_{bcd}$ are identical; for other classes, where $P^a{}_{bcd}$ was not identically zero, we were able to obtain some information about how much $K^a{}_{bcd}$ and $R^a{}_{bcd}$ differed.

[We note that the scalar part K of the curvature candidate $K^a{}_{bcd}$ does not enter our considerations since it does not occur in (8) or (9). Therefore a curvature candidate $K^a{}_{bcd}$ whose only nonzero part is the scalar K puts absolutely no constraints on the Riemann tensor $R^a{}_{bcd}$.]

In this paper we have completed this analysis by taking all possible classes of $\underset{0}{K}^a{}_{bcd}$ and $\underset{0}{K}_{ab}$ and for those classes where $P^a{}_{bcd}$ is not identically zero we found out as much as we could about $P^a{}_{bcd}$ and hence about $R^a{}_{bcd}$. This information is given in Tables I–III and discussed in Sections 2 and 3.

In Section 4 we discuss the generic nature of these results and confirm that for generic spacetimes, nontrivial curvature candidates which satisfy the Ricci equations are Riemann tensors.

2. THE TABLES

As in Edgar (1990), we use the well-known Petrov (1969) classification scheme, first for $\underset{0}{K}^a{}_{bcd}$, and later for $\underset{0}{P}^a{}_{bcd}$ and $\underset{0}{R}^a{}_{bcd}$. Also, as in Edgar (1990), we classify $\underset{0}{K}_{ab}$, and later $\underset{0}{P}_{ab}$ and $\underset{0}{R}_{ab}$ by means of a Petrov classification of their respective Plebanski tensors $\mathscr{K}$, $\mathscr{P}$, and $\mathscr{R}$ (Plebanski, 1964). The Plebanski tensor and associated classification scheme have been written in NP notation (McIntosh *et al.*, 1981) and have already proved useful in investigating algebraic constraints on the Riemann tensor (McIntosh and Halford, 1982). We use the standard NP symbols (Newman and Penrose, 1962) $\Psi_0, \Psi_1, \ldots, \Phi_{00}, \Phi_{01}, \ldots, \Lambda$ for the tetrad components of $\underset{0}{R}^a{}_{bcd}$, $\underset{0}{R}_{ab}$, and R, respectively; analogous symbols $\hat{\Psi}_0$, $\hat{\Psi}_1, \ldots, \hat{\Phi}_{00}$, $\hat{\Phi}_{01}, \ldots, \hat{\Lambda}$ for the tetrad components of $\underset{0}{P}^a{}_{bcd}$, $\underset{0}{P}_{ab}$, and P, respectively; and $\tilde{\Psi}_0$, $\tilde{\Psi}_1, \ldots, \tilde{\Phi}_{00}$, $\tilde{\Phi}_{01}, \ldots, \tilde{\Lambda}$ for the tetrad components of $\underset{0}{K}^a{}_{bcd}$, $\underset{0}{K}_{ab}$, and K, respectively.

The results are presented in Tables I–III. In Table I we present the restrictions which (9) imposes on $P^a{}_{bcd}$ for all possible Petrov types of $\underset{0}{K}^a{}_{bcd}$. In Table II we present the restrictions which (8) imposes on $P^a{}_{bcd}$ for all possible Petrov types of $\mathscr{K}$. In Table III we combine the results from Tables I and II to list those spaces where $R^a{}_{bcd}$ is not completely determined by $K^a{}_{bcd}$ and to give what information can be deduced about these spaces.

The following points should be noted about the tables:

(a) The classification of the Plebanski tensor in NP notation according to Petrov type, as quoted here from McIntosh *et al.* (1981), corresponds directly to the classification by Segré characteristics (Hall, 1976) and to the Plebanski (1964) scheme. We have not added these extra columns since they are listed in McIntosh *et al.* (1981).

(b) For those types of $\underset{0}{K}^a{}_{bcd}$ (and $\mathscr{K}$) where (9) [and (8)] have only the trivial solution for $P^a{}_{bcd}$, denoted by a dash in Table I (and II), $R^a{}_{bcd}$ is equivalent to $K^a{}_{bcd}$ and this is denoted by $\tilde{\Psi}=\Psi$ ($\tilde{\Phi}=\Phi$) in the last three columns for each of these types in Table I (and II).

(c) In Table II the information in the first six columns is taken directly from Table I in McIntosh and Halford (1982). [Equation (8), which imposes constraints on the tensor $P^a{}_{bcd}$ by the symmetric second-order trace-free tensor K_{ab}, has exactly the same form as equation

$$x_{ic}R^i{}_{bef}+x_{ib}R^i{}_{cef}=0 \tag{10}$$

considered in McIntosh and Halford (1982). There the constraints imposed on $R^a{}_{bcd}$ by the symmetric second-order tensor x_{ab} were considered for the various classes of the Plebanski tensor $\mathscr{P}_x$—the Plebanski tensor formed from the trace-free part of x_{ab} as in equation (10). The results in Table I in reference McIntosh and Halford (1982) can be immediately applied to equation (8) in this paper.] However, there are a few small changes—one minor correction, and a refinement of classification for the Petrov I classes using the new subclasses introduced in McIntosh and Arianrhood (1990); the latter are significant when we discuss the generic nature of our results in Section 4. When the Petrov type of $\mathscr{K}$ is O_2, the Petrov type of $\mathscr{P}$ is D—as can be confirmed by substituting the components of $\hat{\Phi}_{AB}$ in the Plebanski tensor components in (A4) of the Appendix. When the Petrov types of $\mathscr{K}$ are O_{a1} and O_{a2}, the Petrov types of $\mathscr{P}$ and $\underset{0}{P}^a{}_{bcd}$ respectively belong to a distinct subclass $I(M^{\pm})$ of Petrov type I. This subclass of Petrov type I corresponds to the expression $M=(I^3/J^2-6)$ formed from the algebraic invariants of the respective tensor being real. [In fact this subclass, which

Table I. Solutions of (9) for $P^a{}_{bcd}$ According to Petrov Class of $\underset{0}{K}{}^a{}_{bcd}$

Petrov type of $\underset{0}{K}{}^a{}_{bcd}$	Canonical set of nonzero $\tilde{\Psi}_A$	Undetermined $\hat{\Psi}_A$, $\hat{\Phi}_{AB}$, $\hat{\Lambda}$ (all others zero)	Dimension bivector space for $\Theta^a{}_b$	Petrov type of $\underset{0}{P}{}^a{}_{bcd}$	Petrov type of $\mathscr{P}$	Undetermined Ψ_A (all others zero)	Petrov type of $\underset{0}{R}{}^a{}_{bcd}$	Petrov type of $\mathscr{R}$
I	$\tilde{\Psi}_0=\tilde{\Psi}_4$, $\tilde{\Psi}_2$	—		—	—	$\tilde{\Psi}=\Psi$	$\tilde{\Psi}=\Psi$	$\tilde{\Phi}=\Phi$
II	$\tilde{\Psi}_2$, $\tilde{\Psi}_4$	—		—	—	$\tilde{\Psi}=\Psi$	$\tilde{\Psi}=\Psi$	$\tilde{\Phi}=\Phi$
III	$\tilde{\Psi}_3$	—		—	—	$\tilde{\Psi}=\Psi$	$\tilde{\Psi}=\Psi$	$\tilde{\Phi}=\Phi$
D	$\tilde{\Psi}_2$	$\hat{\Psi}_2=-2\hat{\Lambda}$, $\hat{\Phi}_{11}$	2	D	D_{a3}	Ψ_2	D	?
N	$\tilde{\Psi}_4$	$\hat{\Psi}_4$, $\hat{\Phi}_{22}$	2	N	O_2	Ψ_4	N	?
O	—	Any	6	I	I	Any	I	?

Table II. Solutions of (8) for $P^a{}_{bcd}$ According to Petrov Class of $\mathscr{K}$

Petrov type of $\mathscr{K}$	Canonical set of nonzero $\tilde\Phi_{AB}$	Undetermined $\hat\Psi_A$, $\hat\Phi_{AB}$, $\hat\Lambda$ (all others zero)	Dimension bivector space for $\Theta^a{}_b$	Petrov type of $\underset{0}{P}{}^a{}_{bcd}$	Petrov type of $\mathscr{P}$	Undetermined Φ_{AB}, Λ (all others zero)	Petrov type of $\mathscr{R}$	Petrov type of $\underset{0}{R}{}^a{}_{bcd}$
I_a	$\tilde\Phi_{00}=\tilde\Phi_{22}$, $\tilde\Phi_{02}=\tilde\Phi_{20}$, $\tilde\Phi_{11}$	—		—	—	$\tilde\psi=\psi$	$\tilde\phi=\phi$	$\tilde\psi=\psi$
D_{a1}	$\tilde\Phi_{02}=\tilde\Phi_{20}$, $\tilde\Phi_{11}$	$\hat\Psi_2=-2\hat\Lambda$, $\hat\Phi_{11}=-3\hat\Lambda$	1	D	D_{a3}	$\Phi_{02}=\Phi_{20}$, Φ_{11}, Λ	D_{a1}	?
D_{a2}	$\tilde\Phi_{00}=\tilde\Phi_{22}$, $\tilde\Phi_{11}$	$\hat\Psi_2=-2\hat\Lambda$, $\hat\Phi_{11}=3\hat\Lambda$	1	D	D_{a3}	$\Phi_{00}=\Phi_{22}$, Φ_{11}, Λ	D_{a2}	?
D_{a3}	$\tilde\Phi_{11}$	$\hat\Psi_2=-2\hat\Lambda$, $\hat\Phi_{11}$	2	D	D_{a3}	Φ_{11}, Λ	D_{a3}	?
O_{a1}	$\tilde\Phi_{02}=\tilde\Phi_{20}=-2\tilde\Phi_{11}$	$\hat\Psi_0=\hat\Phi_{00}$, $\hat\Psi_4=\hat\Phi_{22}$, $\hat\Psi_2+2\hat\Lambda=\hat\Phi_{02}=\hat\Phi_{20}$, $\hat\Psi_2=\hat{\hat\Phi}_{11}+\hat\Lambda$, $\hat\Psi_1=\hat{\hat\Psi}_1=\hat\Phi_{01}$, $\hat\Psi_3=\hat{\hat\Psi}_3=\hat\Phi_{21}$	3	$I(M^\pm),\ldots$	$I(M^\pm),\ldots$	$\Phi_{02}=\Phi_{20}$, Φ_{00}, Φ_{22}, Φ_{11}, Φ_{12}, Φ_{01}, Λ	$I(M^\pm)$	?
O_{a2}	$\tilde\Phi_{00}=\tilde\Phi_{22}=2\tilde\Phi_{11}$	$\hat\Psi_0=\hat\Psi_4=\hat\Phi_{02}$, $\hat\Psi_2+2\hat\Lambda=\hat\Phi_{00}=\hat\Phi_{22}$, $\hat\Psi_2=-\hat{\hat\Phi}_{11}+\hat\Lambda$, $\hat\Psi_1=-\hat{\hat\Psi}_3=-\hat\Phi_{01}=\hat\Phi_{12}$	3	$I(M^\pm),\ldots$	$I(M^\pm),\ldots$	$\Phi_{00}=\Phi_{22}$, Φ_{02}, Φ_{11}, $\Phi_{01}=-\Phi_{12}$, Λ	$I(M^\pm)$	?
O_{a3}	—	Any	6	$I,\ldots$	$I,\ldots$	Any	I	?
I_b	$\tilde\Phi_{00}=-\tilde\Phi_{22}$, $\tilde\Phi_{02}=\tilde\Phi_{20}$, $\tilde\Phi_{11}$	—		—	—	$\tilde\Psi=\Psi$	$\tilde\Phi=\Phi$	$\tilde\Psi=\Psi$
D_b	$\tilde\Phi_{00}=-\tilde\Phi_{22}$, $\tilde\Phi_{11}$	$\hat\Psi_2=-2\hat\Lambda$, $\hat\Phi_{11}=3\hat\Lambda$	1	D	D_{a3}	$\Phi_{00}=-\Phi_{22}$, Φ_{11}, Λ	D_b	?
II	$\tilde\Phi_{02}=\tilde\Phi_{20}$, $\tilde\Phi_{11}$, $\tilde\Phi_{22}$	—		—	—	$\tilde\Psi=\Psi$	$\tilde\Phi=\Phi$	$\tilde\Psi=\Psi$
D_2	$\tilde\Phi_{11}$, $\tilde\Phi_{22}$	$\hat\Psi_2=-2\hat\Lambda$, $\hat\Phi_{11}=3\hat\Lambda$	1	D	D_{a3}	Φ_{11}, Φ_{22}, Λ	D_2	?
N_2	$\tilde\Phi_{02}=\tilde\Phi_{20}$ $=-2\tilde\Phi_{11}$, $\tilde\Phi_{22}$	$\hat\Psi_4=\hat\Phi_{22}$	1	N	O_2	$\Phi_{02}=\Phi_{20}$ $=-2\Phi_{11}$, Φ_{22}, Λ	N_2	?
O_2	$\tilde\Phi_{22}$	$\hat\Psi_4$, $\hat\Psi_3=-\hat\Phi_{21}$, $\hat\Psi_2=-2\hat\Lambda$, $\hat\Phi_{11}=3\hat\Lambda$, $\hat\Phi_{22}$	3	$II,\ldots$	$D,\ldots$	Φ_{22}, Φ_{21}, Φ_{11}, Λ	D	?
III	$\tilde\Phi_{02}=\tilde\Phi_{20}=-2\tilde\Phi_{11}$, $\tilde\Phi_{12}\neq\tilde\Phi_{21}$	—		—	—	$\tilde\Psi=\Psi$	$\tilde\Phi=\Phi$	$\tilde\Psi=\Psi$
N_3	$\tilde\Phi_{12}=-\tilde\Phi_{21}$	$\hat\Psi_4=\hat\Phi_{22}$	1	N	O_2	$\Phi_{12}=-\Phi_{21}$, Φ_{22}, Λ	N_2	?

Table III. Petrov Types of $\underset{0}{K}{}^a{}_{bcd}$ and $\mathscr{K}$ Which Permit Nontrivial Solutions (8) and (9), and Petrov Types of $\underset{0}{R}{}^a{}_{bcd}$ and $\mathscr{R}$ Which Permit Nontrivial Solutions of (8) and (9)

Petrov type of $\underset{0}{K}{}^a{}_{bcd}$	Petrov type of $\mathscr{K}$	Dimension bivector space for $\Theta^a{}_b$	Petrov type of $\underset{0}{P}{}^a{}_{bcd}$	Petrov type of $\mathscr{P}$	Petrov type of $\underset{0}{R}{}^a{}_{bcd}$	Petrov type of $\mathscr{R}$
D	D	2	D	D	D	D
D	O	2	D	D	D	D
N	N	1	N	O	N	N
N	O	1	N	O	N	N
O	D	2	D	D	D	D
O	N	1	N	O	N	N
O	O^*	3	$I(M^{\pm}), II$	$I(M^{\pm}), D$	$I(M^{\pm}), II$	$I(M^{\pm}), D$
O	O_{a3}	6	Any	Any	Any	Any

we have called $I(M^{\pm})$, is really the union of the two subclasses $I(M^{+})$ and $I(M^{-})$ in the classification scheme discussed in McIntosh and Arianrhood (1990).]

(d) For each of those types of $\underset{0}{K}{}^a{}_{bcd}$ (and $\mathscr{K}$) where equations (9) [and (8)] have a nontrivial solution for $\underset{0}{P}{}^a{}_{bcd}$, all undetermined $\hat{\Psi}_A$, $\hat{\Phi}_{AB}$, and $\hat{\Lambda}$ are listed in column 3 of Table I (and II). The Petrov types of $\underset{0}{P}{}^a{}_{bcd}$ and $\mathscr{P}$ listed in columns 5 and 6, respectively, of Table I (and II) are the most general possible—assuming all undetermined $\hat{\Psi}_A$, $\hat{\Phi}_{AB}$, and $\hat{\Lambda}$ in column 3 of Table I (and II) are nonzero and $\hat{\Psi}_A$ and $\hat{\Phi}_{AB}$ have no additional relations between them. For each of these types in Table I and most of these types in Table II the most general Petrov types of $\underset{0}{P}{}^a{}_{bcd}$ and $\mathscr{P}$ are the *only* types—except for $\underset{0}{P}{}^a{}_{bcd}$ and $\mathscr{P}$ being identically zero. However, in Table II, when $\mathscr{K}$ is Petrov type O, clearly some of the undetermined $\hat{\Psi}_A$, $\hat{\Phi}_{AB}$, and $\hat{\Lambda}$ in column 3 could be chosen zero or with additional relations between them, permitting nontrivial subtypes of $\underset{0}{P}{}^a{}_{bcd}$ and $\mathscr{P}$; this feature is denoted by $I(M^{\pm}), \ldots$ and $II, \ldots$ in columns 5 and 6 of Table II.

(e) All tables list the dimension of the bivector space which spans the curvature 2-form $\Theta^a{}_b$ denoted by

$$\Theta^a{}_b = \tfrac{1}{2} P^a{}_{bcd}\, \theta^c \wedge \theta_d \tag{11}$$

where

$$ds^2 = g_{ab}\theta^a\theta^b \tag{12}$$

The $\Theta^a{}_b$ are written out explicitly, in NP notation, in terms of a basis in equations (5.4) of McIntosh and Halford (1982); the dimension can easily be found by substitution in these equations.

(f) For each of those types of $\underset{0}{K}{}^a{}_{bcd}$ (and $\mathscr{K}$) where equation (9) [and (8)] has a nontrivial solution for $\underset{0}{P}{}^a{}_{bcd}$, information on $\underset{0}{R}{}^a{}_{bcd}$ and $\mathscr{R}$ is listed in columns 7–9 of Table I (and II). In Table I the undetermined values of Ψ_A listed in column 7 are obtained by adding each $\tilde{\Psi}_A$ to the corresponding $\hat{\Psi}_A$ listed respectively in columns 2 and 3; therefore the most general Petrov type of $\underset{0}{R}{}^a{}_{bcd}$ can easily be determined in each of the two nontrivial cases, and is listed in column 8; but no information is known directly about $\mathscr{R}$ (until $\mathscr{K}$ is known) and this is denoted by question marks in column 9. In Table II the undetermined values of Φ_{AB} listed in column 8 are obtained by adding each $\tilde{\Phi}_{AB}$ to the corresponding $\hat{\Phi}_{AB}$ listed respectively in columns 2 and 3; therefore the most general Petrov type of $\mathscr{R}$ can be determined and is listed in column 8, but no information is known directly about $\underset{0}{R}{}^a{}_{bcd}$ (until $\underset{0}{K}{}^a{}_{bcd}$ is known) and this is denoted by question marks in column 9. The "most general" Petrov types of $\underset{0}{R}{}^a{}_{bcd}$ listed in column 8 of Table I are the only possible types (except $\underset{0}{R}{}^a{}_{bcd}$ identically zero, when the $\tilde{\Psi}_A$'s and $\hat{\Psi}_A$'s "cancel"), but almost all of the "most general" Petrov types of $\mathscr{K}$ listed in column 8 of Table II have nontrivial subtypes; these can be obtained by choosing the permitted nontrivial subtypes (where possible) or trivial subtypes of $\underset{0}{P}{}^a{}_{bcd}$ and/or $\mathscr{P}$ or by choosing some of the $\hat{\Phi}_{AB}$'s to cancel corresponding $\tilde{\Phi}_{AB}$'s.

(g) The choice of tetrad which gives the canonical form for a particular Petrov type of $\underset{0}{K}{}^a{}_{bcd}$ is not of course the same choice of tetrad which puts a particular Petrov type of $\mathscr{K}$ into canonical form. Therefore, in order to decide which nontrivial solutions of $P^a{}_{bcd}$ in Table I are compatible with nontrivial solutions of $P^a{}_{bcd}$ in Table II we cannot compare directly the undetermined $\hat{\Psi}_A$, $\hat{\Phi}_{AB}$, and $\hat{\Lambda}$ in column 3 of the two tables, but rather compare their invariant Petrov types from the two tables, and from this we build up in Table III the complete picture of those classes of $K^a{}_{bcd}$ which permit nontrivial solutions of $P^a{}_{bcd}$. For the same reason, for these classes, when we are trying to find information on the Riemann tensor we cannot combine directly the results obtained for Ψ_A in column 7 of Table I with the results obtained for Φ_{AB} and Λ in column 7 of Table II—rather, we must use the Petrov types to obtain the information in columns 6 and 7 of Table III. We demonstrate how we construct Table III in the Appendix.

(h) The Petrov types of $\underset{0}{R}{}^a{}_{bcd}$ and $\mathscr{R}$ listed in columns 6 and 7 of Table III are the most general, and in most cases nontrivial subtypes can be found, as is clear from the discussion in the Appendix.

(i) In Table III class O^* means types O_{a1}, O_{a2}, O_2. The O_{a3} type—which is when all $\tilde{\Phi}_{AB}$ are zero, i.e., $\underset{0}{K}_{ab}=0$—is listed separately since for this class (with only the scalar $\tilde{\Lambda}$ nonzero) $K^a{}_{bcd}$ imposes no conditions on $P^a{}_{bcd}$ nor therefore on $R^a{}_{bcd}$.

3. SUMMARY OF RESULTS

We highlight the following results from Table III.

(i) Only the trivial solution of (6) [equivalently (8) and (9)] is possible when either of the following conditions on $K^a{}_{bcd}$ holds:

(a) $\underset{0}{K}{}^a{}_{bcd}$ is Petrov type *I*, *II*, or *III*.
(b) $\mathscr{K}$ is Petrov type *I*, *II*, or *III*.
(c) $\underset{0}{K}{}^a{}_{bcd}$ is Petrov type *D* and $\mathscr{K}$ is Petrov type *N*.
(d) $\underset{0}{K}{}^a{}_{bcd}$ is Petrov type *N* and $\mathscr{K}$ is Petrov type *D*.

This result was given in Edgar (1990).

(ii) Nontrivial solutions of (6) only occur when both $\underset{0}{K}{}^a{}_{bcd}$ and $\mathscr{K}$ are degenerate Petrov classes (i.e., *D*, *N*, or *O*). However, for some of these cases $K^a{}_{bcd}$ may be of rank 6—the maximal rank. This can easily be seen by substituting these classes of $K^a{}_{bcd}$ into the equivalent equation to (5.4) in McIntosh and Halford (1982), for $K^a{}_{bcd}$.

(iii) When nontrivial solutions of (6) occur the tensor $P^a{}_{bcd}$, which measures the difference between $K^a{}_{bcd}$ and $R^a{}_{bcd}$, has dimension at most 3. Both $\underset{0}{P}{}^a{}_{bcd}$ and $\mathscr{P}$ are degenerate Petrov classes [i.e., $I(M^{\pm})$, II, D, N, or O].

(iv) Nontrivial solutions of (6) only occur when both $\underset{0}{R}{}^a{}_{bcd}$ and $\mathscr{R}$ are degenerate Petrov classes [i.e., $I(M^{\pm})$, II, D, N, or O]. For some of these cases $R^a{}_{bcd}$ may be of rank 6—the maximal rank. However, we note that a spacetime with a high-ranking Riemann tensor [and $\underset{0}{R}{}^a{}_{bcd}$ and $\mathscr{R}$ both Petrov type $I(M^{\pm})$] only permits nontrivial solutions in one very special situation—when *both* $\underset{0}{K}{}^a{}_{bcd}$ and $\mathscr{K}$ are type O.

So it is only in very special situations that the Ricci equations are insufficient to characterize a Riemann tensor; it is only when the $\underset{0}{K}{}^a{}_{bcd}$ and $\mathscr{K}$ parts of $K^a{}_{bcd}$ are both degenerate, or alternatively when a spacetime has a Riemann tensor $R^a{}_{bcd}$ which has both $\underset{0}{R}{}^a{}_{bcd}$ and $\mathscr{R}$ parts degenerate. Of course in the latter case we are assuming that the curvature candidate $K^a{}_{bcd}$ has more structure than just its scalar curvature K, since we already pointed out in Section 1 that such a curvature candidate puts no constraint on $R^a{}_{bcd}$.

4. GENERIC NATURE OF RESULTS

It has been noted that it is only for spacetimes whose Riemann tensors are very specialized that the Ricci equations fail to characterize the Riemann tensor. So we would suspect that this statement can formally be made for "generic" spacetimes.

Referring to Rendall (1988*a*), we find from Proposition 6 : 3 that there exists an open dense set of C^r Lorentz metrics on **M** with Whitney C^r $(r \geq 3)$ topology whose Weyl tensors are Petrov type I at all points of the spacetime **M**, except possibly on a two-dimensional regular submanifold of **M**, where the type is II, and isolated points where it is III or D.

When we apply this proposition to Table III we can conclude that for a generic spacetime, a curvature candidate $K^a{}_{bcd}$—with at least one of $\underset{0}{K}{}^a{}_{bcd}$ and $\mathscr{K}$ not Petrov type O—which satisfies the Ricci equations is equal to a Riemann tensor at all points of **M** (except perhaps at isolated points.)

However, we can even strengthen this result by weakening the condition on $K^a{}_{bcd}$. First we need to strengthen Proposition 6 : 3 by replacing *Petrov type I* with that general subclass of Petrov type I which excludes Petrov type $I(M^{\pm})$; this subclass, which corresponds to **M** being complex, will be labeled *Petrov type* $I(M^C)$. So we need to construct a slightly finer stratification where Petrov type I is subdivided into types $I(M^{\pm})$ and $I(M^C)$, but the rest of the stratification is unchanged. By the same type of argument as used by

Rendall (1987) to establish Proposition 6 : 3,[2] we can show that there does exist an open dense subset of C^r Lorentz metrics on **M** whose Weyl tensors are Petrov type $I(M^C)$; of course there is the possibility that it can reduce to Petrov type $I(M^{\pm})$ on a submanifold.

Therefore we can conclude that for a generic spacetime, a curvature candidate $K^a{}_{bcd}$—with at least one of $\underset{0}{K}{}^a{}_{bcd}$ and $\mathscr{K}$ nonzero—which satisfies the Ricci equations is equal to a Riemann tensor (except possibily on some submanifolds of **M**).

APPENDIX

$\underset{0}{K}{}^a{}_{bcd}$ Petrov Type D

From Table I it follows immediately that $\underset{0}{P}{}^a{}_{bcd}$ is type D, $\mathscr{P}$ is type D_{a3}, and $\hat{\Lambda} \neq 0$.

(a) *If* $\mathscr{K}$ *is type* D, then it follows immediately from Table II that $\underset{0}{P}{}^a{}_{bcd}$ is type D, $\mathscr{P}$ is Type D_{a3}, and $\hat{\Lambda} \neq 0$. This result coincides exactly with the result given above from Table I. So for this class of curvature candidate $R^a{}_{bcd}$ differs from $K^a{}_{bcd}$. However, from Table I we also note that $\underset{0}{R}{}^a{}_{bcd}$ is type D and from Table II, $\mathscr{R}$ is type D, i.e., for this class of curvature candidate although $R^a{}_{bcd}$ may differ from $K^a{}_{bcd}$, it has the same Petrov types, and the scalars K and R may differ.

(b) *If* $\mathscr{K}$ *is type* N, then it follows immediately from Table II that $\underset{0}{P}{}^a{}_{bcd}$ is type N, $\mathscr{P}$ is type O_2 and $\hat{\Lambda} = 0$. These results can only be reconciled with those given above from Table I when both $\underset{0}{P}{}^a{}_{bcd}$ and $\mathscr{P}$ as well as $\hat{\Lambda}$ are identically zero, i.e., for this class of curvature candidate $K^a{}_{bcd}$ is identical to $R^a{}_{bcd}$.

(c) *If* $\mathscr{K}$ *is type* O, then it follows immediately from Table II that $\underset{0}{P}{}^a{}_{bcd}$ is type $I, \ldots$ or $II, \ldots$, $\mathscr{P}$ is type $I, \ldots$ or $II, \ldots$, and $\hat{\Lambda} \neq 0$. There will be a nontrivial intersection between the results given above from Table I and these results from Table II provided that, in the latter, type D is a permitted subtype for $\underset{0}{P}{}^a{}_{bcd}$ and type D_{a3} is a permitted subtype for $\mathscr{P}$. Of course the tetrad frame in which $P^a{}_{bcd}$ is given in Table I (the usual canonical tetrad frame associated with the different Petrov types of $\mathscr{K}$ given in Table II, so a direct comparison between column 3 in each table is not sufficient; however, it is easy to deduce that the required subtypes do exist, although we cannot immediately obtain their most general form in the tetrad in which $\mathscr{K}$ has its canonical form. So when $\underset{0}{K}{}^a{}_{bcd}$ is type D and $\mathscr{K}$ is type O, $R^a{}_{bcd}$ differs from $K^a{}_{bcd}$.

[2] I am grateful to Dr. Rendall for help on this point.

From Table I we also note that $\underset{0}{R}{}^{a}{}_{bcd}$ is type D and from Table II that $\mathscr{R}$ is type $I, \ldots$. However, we can be more precise in our prediction for $\mathscr{R}$ when we make $P^{a}{}_{bcd}$ in Table II compatible with $P^{a}{}_{bcd}$ in Table I. If we transform $P^{a}{}_{bcd}$ given in Table I into an arbitrary tetrad frame, the $\hat{\Psi}_A$'s have the form (Ludwig, 1986)

$$\begin{aligned}
\hat{\Psi}_0 &= 6a^2b^2\hat{\Psi} \\
\hat{\Psi}_1 &= 3ab(ad+bc)\hat{\Psi} \\
\hat{\Psi}_2 &= (a^2d^2+b^2c^2+4abcd)\hat{\Psi} \\
\hat{\Psi}_3 &= 3cd(ad+bc)\hat{\Psi} \\
\hat{\Psi}_4 &= 6c^2d^2\hat{\Psi}
\end{aligned} \tag{A1}$$

and the $\hat{\Phi}_{AB}$'s have the form

$$\begin{aligned}
\hat{\Phi}_{00} &= 4ab\overline{ab}\hat{\Phi} \\
\hat{\Phi}_{01} &= 2ab\overline{(ad+bc)}\hat{\Phi} \\
\hat{\Phi}_{02} &= 4ab\overline{cd}\hat{\Phi} \\
\hat{\Phi}_{11} &= (ad+bc)\overline{(ad+bc)}\hat{\Phi} \\
\hat{\Phi}_{12} &= 2cd\overline{(ad+bc)}\hat{\Phi} \\
\hat{\Phi}_{22} &= 4cd\overline{cd}\hat{\Phi}
\end{aligned} \tag{A2}$$

where both $\hat{\Psi}$ and $\hat{\Phi}$ are real, corresponding to the $\hat{\Psi}_2$ and $\hat{\Phi}_{11}$, respectively in Table I. Substituting in column 3 of Table II gives the conditions under which these are consistent with the form of the undetermined $\hat{\Psi}_A$'s and $\hat{\Phi}_{AB}$'s when $\mathscr{K}$ is type O_{a1}, O_{a2}, or O_2, respectively. (Of course, when $\mathscr{K}$ is type O_{a3} there is always consistency.)

When $\mathscr{K}$ is type O_{a1} the above equations are found to be consistent providing

$$\begin{aligned}
ab &= \overline{ab} \\
cd &= \overline{cd} \\
(ad+bc) &= \overline{(ad+bc)} \\
2\hat{\Phi} &= 3\hat{\Psi}
\end{aligned} \tag{A3}$$

The Φ_{AB} are found by adding the $\tilde{\Phi}_{AB}$ in column 2 of Table II to the $\hat{\Phi}_{AB}$ given in (A2) with the conditions (A3) substituted. The components of the

Plebanski tensor $\mathscr{R}$ can now be written out, using McIntosh *et al.* (1981),

$$
\begin{aligned}
\chi_0 &= \tfrac{1}{2}(\Phi_{00}\Phi_{02}-\Phi_{01}^2) \\
&= -2a^2b^2(\hat{\Phi}^2+\hat{\Phi}\tilde{\Phi}_{11}) \\
\chi_1 &= \tfrac{1}{4}(\Phi_{00}\Phi_{12}+\Phi_{10}\Phi_{02}-2\Phi_{01}\Phi_{11}) \\
&= -ab(ab+cd)(\hat{\Phi}^2+\hat{\Phi}\tilde{\Phi}_{11}) \\
\chi_2 &= \tfrac{1}{12}(\Phi_{00}\Phi_{22}-4\Phi_{11}^2+4\Phi_{10}\Phi_{12}-2\Phi_{21}\Phi_{01}+\Phi_{02}\Phi_{20}) \\
&= -\tfrac{1}{3}(a^2d^2+b^2c^2+4abcd)(\hat{\Phi}^2+\hat{\Phi}\tilde{\Phi}_{11}) \\
\chi_3 &= \tfrac{1}{4}(\Phi_{22}\Phi_{10}+\Phi_{12}\Phi_{20}-2\Phi_{21}\Phi_{11}) \\
&= -cd(ad+bc)(\hat{\Phi}^2+\hat{\Phi}\tilde{\Phi}_{11}) \\
\chi_4 &= \tfrac{1}{2}(\Phi_{22}\Phi_{20}-\Phi_{21}^2) \\
&= -2c^2d^2(\hat{\Phi}^2+\hat{\Phi}\tilde{\Phi}_{11})
\end{aligned}
\qquad \text{(A4)}
$$

By comparing these with (A1), it is obvious that $\mathscr{R}$ is also Petrov type D.

When a similar analysis is applied to the other two cases—when $\mathscr{K}$ is type O_{a2} and O_2, respectively—it is found for these cases also that $\mathscr{R}$ is Petrov type D.

So for this class of curvature candidate if $\underset{0}{K}{}^a{}_{bcd}$ and $\underset{0}{R}{}^a{}_{bcd}$ differ, they are the same type; if $\mathscr{K}$ and $\mathscr{R}$ differ, $\mathscr{R}$ is at most of type D and the scalars K and R can differ.

Therefore we have obtained the first section of Table III.

$\underset{0}{K}{}^a{}_{bcd}$ Petrov Type N

From Table I it follows immediately that $\underset{0}{P}{}^a{}_{bcd}$ is type N, $\mathscr{P}$ is type O_2 and $\hat{\Lambda}=0$.

(a) *If $\mathscr{K}$ is type D*, then it follows immediately from Table II that $\underset{0}{P}{}^a{}_{bcd}$ is type D, $\mathscr{P}$ is type D_{a3}, and $\hat{\Lambda}\neq 0$. These results can only be reconciled with those from Table I in the previous paragraph when $\underset{0}{P}{}^a{}_{bcd}$, $\mathscr{P}$, and $\hat{\Lambda}$ are all identically zero, i.e., for this class of curvature candidate $K^a{}_{bcd}$ is identical to $R^a{}_{bcd}$.

(b) *If $\mathscr{K}$ is type N*, then it follows immediately from Table II that $\underset{0}{P}{}^a{}_{bcd}$ is type N, $\mathscr{P}$ is type O_2, and $\hat{\Lambda}=0$. This result coincides exactly with the result given above from Table I. So for this class of curvature candidate $R^a{}_{bcd}$ differs from $K^a{}_{bcd}$. However, from Table I we also note that $\underset{0}{R}{}^a{}_{bcd}$ is type N and from Table II, $\mathscr{R}$ is type N, i.e., for this class of curvature candidate, although $R^a{}_{bcd}$ may differ from $K^a{}_{bcd}$, it has the same Petrov types and the scalars K and R are equal.

(c) *If $\mathscr{K}$ is type O*, then it follows immediately from Table II that $\underset{0}{P}{}^{a}{}_{bcd}$ is type *I*, ... or *II*, ..., $\mathscr{P}$ is type *I*, ... or *II*, ..., and $\hat{\Lambda} \neq 0$. There will be a nontrivial intersection for $P^{a}{}_{bcd}$ between the results given from Table I at the beginning of this subsection and these results from Table II provided that, in the latter, type *N* is a permitted subtype for $\underset{0}{P}{}^{a}{}_{bcd}$ and type O_2 is a permitted subtype for $\mathscr{P}$ when $\hat{\Lambda} = 0$. It is easy to deduce that the required subtypes are permitted—at least when $\mathscr{K}$ is type O_{a1}, O_{a3} (trivially), or O_2—although we cannot immediately obtain their most general form in the tetrad in which $\mathscr{K}$ has its canonical form. So when $\underset{0}{K}{}^{a}{}_{bcd}$ is type *N* and $\mathscr{K}$ is type *O*, $R^{a}{}_{bcd}$ differs from $K^{a}{}_{bcd}$.

From Table I we note that, for these types, $\underset{0}{R}{}^{a}{}_{bcd}$ is type *N* and from Table II that $\mathscr{R}$ is type *I*, ... However, we can be more precise in our prediction for $\mathscr{R}$ by using Table I and noting that it is obtained from the Φ_{AB}'s which are found by adding $\bar{\Phi}_{22}$ to the type *O* $\tilde{\Phi}_{AB}$'s (which are not of course necessarily in canonical form). Such a minor change is easily seen [by considering explicitly the tetrad components of the Plebanski tensor $\mathscr{R}$ in (A4)] to ensure that the most general $\mathscr{R}$ is type *N*. So for this class of curvature candidate if $\underset{0}{K}{}^{a}{}_{bcd}$ and $\underset{0}{R}{}^{a}{}_{bcd}$ differ, they are the same type; if $\mathscr{K}$ and $\mathscr{R}$ differ, $\mathscr{R}$ is at most of type *N* and the scalars *K* and *R* are equal.

Therefore we have obtained the middle section of Table III.

$\underset{0}{K}{}^{a}{}_{bcd}$ Petrov Type O

When $\underset{0}{K}{}^{a}{}_{bcd}$ is type *O*, i.e., identically zero, then we can use Table II directly, combined with the fact that $\underset{0}{R}{}^{a}{}_{bcd}$ is identical to $\underset{0}{P}{}^{a}{}_{bcd}$. So we find from Table II that the only nontrivial solutions for $P^{a}{}_{bcd}$ are when $\mathscr{K}$ is type *D*, *N*, or *O*. This enables us to obtain the last three lines of Table III. We have separated out the case when $\mathscr{K}$ is type O_{a3} since for this class both $\underset{0}{K}{}^{a}{}_{bcd}$ and $\underset{0}{K}{}_{ab}$ are identically zero.

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